

# 1 Relevant First-Order Logic $LP^\#$ and Curry's Paradox

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**Abstract:** In this article I will sketch the structure of non-classical approach to the well known Curry's Paradox.

## 2 Introduction.

In this article I will sketch the structure of non-classical approaches to the well known Curry's Paradox. Similar approach are used in my previous works [1],[2]. One approach to the paradoxes of self-reference takes these paradoxes as motivating a non-classical theory of logical consequence. Similar logical principles are used in each of the paradoxical inferences. If one or other of these problematic inferences are rejected, we may arrive at a consistent (or at least, a coherent) theory. The general approach of using the paradoxes to restrict the class of allowable inferences places severe constraints on the domain of possible propositional logics, and on the kind of metatheory that is appropriate in the study of logic itself.

## 3 I. Relevant First-Order Logics in General

Relevance logics are non-classical logics. Called 'relevant logics' in Britain and Australasia, these systems developed as attempts to avoid the paradoxes of material and strict implication. It is well known that relevant logic does not accept an axiom scheme  $A \rightarrow (\neg A \rightarrow B)$  and the rule  $A, \neg A \vdash B$ . (cf. [Routley et al 1983]). Hence, in a natural way it might be used as basis for contradictory but non-trivial theories, i.e. paraconsistent ones. Among the paradoxes of material implication are:  $p \rightarrow (q \rightarrow p)$ ,  $\neg p \rightarrow (p \rightarrow q)$ ,  $(p \rightarrow q) \vee (q \rightarrow r)$ . Among the paradoxes of strict implication are the following:  $(p \wedge \neg p) \rightarrow q$ ,  $p \rightarrow (q \rightarrow q)$ ,  $p \rightarrow (q \neg q)$ .

Relevant logicians point out that what is wrong with some of the paradoxes (and fallacies) is that is that the antecedents and consequents (or premises and

conclusions) are on completely different topics. The notion of a topic, however, would seem not to be something that a logician should be interested in — it has to do with the content, not the form, of a sentence or inference. But there is a formal principle that relevant logicians apply to force theorems and inferences to “stay on topic”. This is the variable sharing principle. The variable sharing principle says that no formula of the form  $A \rightarrow B$  can be proven in a relevance logic if  $A$  and  $B$  do not have at least one propositional variable (sometimes called a proposition letter) in common and that no inference can be shown valid if the premises and conclusion do not share at least one propositional variable.

In the work of Anderson and Belnap the central systems of relevance logic were the logic **E** of relevant entailment and the system **R** of relevant implication. The relationship between the two systems is that the entailment connective of **E** was supposed to be a strict (i.e. necessitated) relevant implication. To compare the two, Meyer added a necessity operator to **R** (to produce the logic **NR**). Larisa Maksimova, however, discovered that **NR** and **E** are importantly different — that there are theorems of **NR** (on the natural translation) that are not theorems of **E**. This has left some relevant logicians with a quandary. They have to decide whether to take **NR** to be the system of strict relevant implication, or to claim that **NR** was somehow deficient and that **E** stands as the system of strict relevant implication. (Of course, they can accept both systems and claim that **E** and **R** have a different relationship to one another.)

## 4 II. Curry’s Paradox

The paradoxes of self-reference are genuinely paradoxical. The liar paradox, Russell’s paradox and their cousins pose enormous difficulties to anyone who seeks to give a comprehensive theory of semantics, or of sets, or of any other domain which allows a modicum of self-reference and a modest number of logical principles. They have straightforward general features: Firstly, we keep whatever semantic, or set-theoretic principles are at issue. For example, if it is the liar paradox in question, we can keep the naive truth scheme, to the effect that  $\mathbf{T}[\mathbf{A}] \leftrightarrow \mathbf{A}$  where  $[\circ]$  is some name-forming functor, taking sentences to names, and where  $\leftrightarrow$  is some form of biconditional. This scheme says, in effect, that  $\mathbf{T}[\mathbf{A}]$  is true under the same circumstances as  $\mathbf{A}$ . To assert that  $\mathbf{A}$  is true is saying no more and no less than asserting  $\mathbf{A}$ . Secondly, we allow our language to contain a modicum of self-reference. We wish to express sentences such as the liar: “This sentence is not true.” If the language in question is a natural language, then indexicals will do the trick. If the language is a formal language without indexicals, some other technique will be needed to construct sentences analogous to the liar. A Gödel numbering and a means of diagonalisation will do nicely to give the required results.

The paradox I have in mind can be found in a logic independently of its stand on negation. The deduction appeals to no particular principles of negation, as it is negation-free. Any deduction must use some inferential principles.

**Here are the principles needed to derive the paradox.**

**A transitive relation of consequence:** We write this by  $\vdash$ . I take  $\vdash$  to be a relation between statements, and I require that it be transitive: if  $\mathbf{A} \vdash \mathbf{B}$  and  $\mathbf{B} \vdash \mathbf{C}$  then  $\mathbf{A} \vdash \mathbf{C}$ .

**Conjunction and implication:**

I require that the conjunction operator  $\wedge$  be a greatest lower bound with respect to  $\vdash$ . That is,  $\mathbf{A} \vdash \mathbf{B}$  and  $\mathbf{A} \vdash \mathbf{C}$

if and only if  $\mathbf{A} \vdash \mathbf{B} \wedge \mathbf{C}$ .

Furthermore, I require that there be a residual for conjunction: a connective  $\rightarrow$  such that  $\mathbf{A} \wedge \mathbf{B} \vdash \mathbf{C}$  if and only if  $\mathbf{A} \vdash \mathbf{B} \rightarrow \mathbf{C}$ .

**Modus Ponens rule :**  $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash \mathbf{B}$ .

**Modus Tollens rule:**  $\mathbf{P} \rightarrow \mathbf{Q}, \sim \mathbf{Q} \vdash \sim \mathbf{P}$ .

**A paradox generator:** We need only a very weak paradox generator. We take the **T** scheme in the following enthymematic form:

$$\mathbf{T}[\mathbf{A}] \wedge \mathbf{C} \vdash \mathbf{A} \qquad \mathbf{A} \wedge \mathbf{C} \vdash \mathbf{T}[\mathbf{A}]$$

for some true statement **C**. The idea is simple:  $\mathbf{T}[\mathbf{A}]$  need not entail **A**. Take **C** to be the conjunction of all required background constraints.

**Diagonalisation.** To generate the paradox we use a technique of diagonalisation to construct a statement  $\xi$  such that  $\xi$  is equivalent to  $\mathbf{T}[\xi] \rightarrow \mathbf{A}$ , where **A** is any statement you please.

Curry's paradox, so named for its discoverer, namely Haskell B. Curry, is a paradox within the family of so-called paradoxes of self-reference (or paradoxes of circularity). Like the liar paradox (e.g., 'this sentence is false') and Russell's paradox, Curry's paradox challenges familiar naive theories, including naive truth theory (unrestricted **T**-schema) and naive set theory (unrestricted axiom of abstraction), respectively. If one accepts naive truth theory (or naive set theory), then Curry's paradox becomes a direct challenge to one's theory of logical implication or entailment. Unlike the liar and Russell paradoxes Curry's paradox is negation-free; it may be generated irrespective of one's theory of negation.

### 2.1.Truth-Theoretic Version.

Assume that our truth predicate satisfies the following **T**-schema.

**T-Schema:**  $T[A] \leftrightarrow A$ ,

where "[o]" is a name-forming device. Assume, too, that we have the principle called Assertion (also known as *pseudo modus ponens*):

**Assertion:**  $(A \wedge (A \rightarrow B)) \rightarrow B$

By diagonalization, self-reference or the like we can get a sentence  $C$  such that  $C = T[C] \rightarrow F$ ,

where  $F$  is anything you like. (For effect, though, make  $F$  something obviously false.) By an instance of the **T-schema** (" $T[C] \leftrightarrow C$ ") we immediately get:

$$T[C] \leftrightarrow (T[C] \rightarrow F),$$

Again, using the same instance of the **T-Schema**, we can substitute  $C[T, F]$  for  $T[C]$  in the above to get (1).

- (1)  $\vdash C[T, F] \leftrightarrow (C[T, F] \rightarrow F)$  [by **T-schema** and Substitution]
- (2)  $\vdash (C[T, F] \wedge (C[T, F] \rightarrow F)) \rightarrow F$  [by Assertion]
- (3)  $\vdash (C[T, F] \wedge C[T, F]) \rightarrow F$  [by Substitution, from 2]
- (4)  $\vdash C[T, F] \rightarrow F$  [by Equivalence of  $C$  and  $C \wedge C$ , from 3]
- (5)  $\vdash C[T, F]$  [by Modus Ponens, from 1 and 4]
- (6)  $\vdash F$  [by Modus Ponens, from 4 and 5]

Letting  $F$  be anything entailing triviality Curry's paradox quickly 'shows'

that the world is trivial.

## 2.2. Set-Theoretic Version

The same result ensues within naive set theory. Assume, in particular, the

(unrestricted) axiom of abstraction (or comprehension):

**Unrestricted Abstraction:**  $x \in \{x | A(x)\} \leftrightarrow A(x)$ .

Moreover, assume that our conditional,  $\rightarrow$ , satisfies Contraction (as above),

which permits the deduction of  $(s \in s \rightarrow A)$  from  $s \in s \rightarrow (s \in s \rightarrow$

$A)$ .

In the set-theoretic case, let  $C[F] \triangleq \{x | x \in x \rightarrow F\}$ , where  $F$  remains as you

please (but something obviously false, for effect). From here we reason thus:

- (1)  $\vdash x \in C[F] \leftrightarrow (x \in x \rightarrow F)$  [by Unrestricted Abstraction]
- (2)  $\vdash C[F] \in C[F] \leftrightarrow (C[F] \in C[F] \rightarrow F)$  [by Universal Specification, from 1]
- (3)  $\vdash C[F] \in C[F] \rightarrow (C[F] \in C[F] \rightarrow F)$  [by Simplification, from 2]

- (4)  $\vdash \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \rightarrow \mathbf{F}$  [by Contraction, from 3]
- (5)  $\vdash \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}]$  [by A Unrestricted Modus Ponens, from 2 and 4]
- (6)  $\vdash \mathbf{F}$  [by A Unrestricted Modus Ponens, from 4 and 5]

So, coupling Contraction with the naive abstraction schema yields, via Curry's paradox, triviality.

**This is a problem. Our true  $\mathbf{C}[\mathbf{F}]$  entails an arbitrary  $\mathbf{F}$ .**

This inference arises independently of any treatment of negation.

The form of the inference is reasonably well known. It is Curry's paradox,

and it causes a great deal of trouble to any non-classical approach to the paradoxes.

In the next section I show how the tools for Curry's paradox are closer to hand than you might think.

## 5 III.Relevant First-Order Logic $\mathbf{LP}^\#$

In order to avoid the results mentioned in (2.1)-(2.2), one could think of restrictions in initial formulation of the rule Unrestricted Modus Ponens.

The postulates (or their axioms schemata) of propositional logic  $\mathbf{LP}^\#[\mathbf{V}]$  are the following:

### I. Logical postulates:

- (1)  $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A}),$
- (2)  $(\mathbf{A} \rightarrow \mathbf{B}) \rightarrow ((\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow (\mathbf{A} \rightarrow \mathbf{C})),$
- (3)  $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B}),$
- (4)  $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{A},$
- (5)  $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B},$
- (6)  $\mathbf{A} \rightarrow (\mathbf{A} \vee \mathbf{B}),$
- (7)  $\mathbf{B} \rightarrow (\mathbf{A} \vee \mathbf{B}),$
- (8)  $(\mathbf{A} \rightarrow \mathbf{C}) \rightarrow ((\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{A} \vee \mathbf{B} \rightarrow \mathbf{C})),$
- (9)  $\mathbf{A} \vee \neg \mathbf{A},$
- (10)  $\mathbf{B} \rightarrow (\neg \mathbf{B} \rightarrow \mathbf{A}).$

### II. Restricted Modus Ponens rule : $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash_r \mathbf{B}$ iff $\mathbf{A} \notin \mathbf{V}$ .

## 6 IV. Curry's paradox dont valid in Relevant First-Order Logic $\mathbf{LP}^\# [\mathbf{V}]$

4.1. Let us consider Curry's paradox in Relevant First-Order Logic  $\mathbf{LP}^\#$  in set-theoretic version.

Let  $\mathbf{C}[\mathbf{F}] \triangleq \{\mathbf{x} | \mathbf{x} \in \mathbf{x} \rightarrow \mathbf{F}\}$  and  $\alpha[\mathbf{F}]$  is a formula such that:  $\alpha[\mathbf{F}] \leftrightarrow \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}]$ .

Let us denotes by symbol  $\mathbf{V}_\Delta$  the set  $\mathbf{V}_\Delta = \{\alpha[\mathbf{F}] | \mathbf{F} \in \Delta\}$ .

From definition above we obtain the:

**Restricted Modus Ponens rule :**  $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash_r \mathbf{B}$  iff  $\mathbf{A} \notin \mathbf{V}_\Delta$ .

Thus  $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \not\vdash_r \mathbf{B}$  if  $\mathbf{A} \in \mathbf{V}_\Delta$ .

From here we reason thus:

- (1)  $\vdash_r \mathbf{x} \in \mathbf{C}[\mathbf{F}] \leftrightarrow (\mathbf{x} \in \mathbf{x} \rightarrow \mathbf{F})$  [by Unrestricted Abstraction]
- (2)  $\vdash_r \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \leftrightarrow (\mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \rightarrow \mathbf{F})$  [by Universal Specification, from 1]
- (3)  $\vdash_r \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \rightarrow (\mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \rightarrow \mathbf{F})$  [by Simplification, from 2]
- (4)  $\vdash_r \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}] \rightarrow \mathbf{F}$  [by Contraction, from 3]
- (5)  $\vdash_r \mathbf{C}[\mathbf{F}] \in \mathbf{C}[\mathbf{F}]$  [by A Restricted Modus Ponens, from 2 and 4]
- (6)  $\not\vdash_r \mathbf{F}$  [by A Restricted Modus Ponens, from 4 and 5]

4.2. Let us consider Curry's paradox in Relevant First-Order Logic  $\mathbf{LP}^\#$

in truth-theoretic version.

Let  $\mathbf{C}[\mathbf{T}, \mathbf{F}] \triangleq \mathbf{T}[\mathbf{C}] \leftrightarrow (\mathbf{T}[\mathbf{C}] \rightarrow \mathbf{F})$  and  $\alpha[\mathbf{T}, \mathbf{F}]$  is a formula such that:  $\alpha[\mathbf{T}, \mathbf{F}] \leftrightarrow \mathbf{T}[\mathbf{C}]$ .

Let us denotes by symbol  $\mathbf{V}_{\Delta, \Lambda}$  the set  $\mathbf{V}_{\Delta, \Lambda} = \{\alpha[\mathbf{T}, \mathbf{F}] | \mathbf{T} \in \Lambda, \mathbf{F} \in \Delta\}$ .

From definition above we obtain the:

**Restricted Modus Ponens rule :**  $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash_r \mathbf{B}$  iff  $\mathbf{A} \notin \mathbf{V}_{\Delta, \Lambda}$ .

Thus  $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \not\vdash_r \mathbf{B}$  if  $\mathbf{A} \in \mathbf{V}_{\Delta, \Lambda}$ .

- (1)  $\vdash_r \mathbf{C}[\mathbf{T}, \mathbf{F}] \leftrightarrow (\mathbf{C}[\mathbf{T}, \mathbf{F}] \rightarrow \mathbf{F})$  [by  $\mathbf{T}$ -schema and Substitution]
- (2)  $\vdash_r (\mathbf{C}[\mathbf{T}, \mathbf{F}] \wedge (\mathbf{C}[\mathbf{T}, \mathbf{F}] \rightarrow \mathbf{F})) \rightarrow \mathbf{F}$  [by Assertion]
- (3)  $\vdash_r (\mathbf{C}[\mathbf{T}, \mathbf{F}] \wedge \mathbf{C}[\mathbf{T}, \mathbf{F}]) \rightarrow \mathbf{F}$  [by Substitution, from 2]
- (4)  $\vdash_r \mathbf{C}[\mathbf{T}, \mathbf{F}] \rightarrow \mathbf{F}$  [by Equivalence of C and C&C, from 3]
- (5)  $\vdash_r \mathbf{C}[\mathbf{T}, \mathbf{F}]$  [by A Restricted Modus Ponens, from 1 and 4]
- (6)  $\not\vdash_r \mathbf{F}$  [by A Restricted Modus Ponens, from 4 and 5]

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